ELECTRON EMISSION FROM COLD CATHODES IN FIELDS WITH HIGH RATES OF RISE

R. N. DeWitt

Naval Surface Weapons Center
Dahlgren, Virginia 22448

Summary

An expression is derived for the time dependent tunneling probability for electron emission from a cold cathode in an electric field with a high rate of rise. The solution is described, and the results are discussed with respect to the static Fowler-Nordheim results.

Introduction

In deriving the expression for the tunneling probability for electron emission from a cold cathode, one begins with the WKB approximation for the wave function satisfying the time independent Schroedinger wave equation. For barrier penetration, this form is used to describe the wave functions in the three regions of the problem: incident, tunneling, and exit. Applying the connection formulas to the turning points, an expression is obtained (1) for the ratio of the exiting to the incident current and, thereby, for the transmission coefficient. Finally, one assumes that the barrier is high and broad to arrive at the integral expression for the transmission coefficient that forms the starting point for most analyses. In pulse power systems the applied field is usually rising rapidly; hence the barrier width and height are decreasing rapidly, and the problem is non-static.

Analysis

Electrons with an energy E in a metal see a potential barrier at the surface of magnitude V_o. The difference V_o - E is the work function W of the metal. When an electric field of magnitude ϵ is applied to the metal, the potential outside the metal becomes

$$V(X) = V_o - e\,\epsilon\,X$$

where e is the electron charge. If the electric field is increasing at a constant rate ϵ_r, then the slope of the potential outside the metal is decreasing steadily with time and the barrier width L is likewise decreasing, as shown in Figure 1.

The standard procedure in obtaining the transmission coefficient is to apply the boundary conditions to the solutions of the Schroedinger equation in the three regions. The solutions in region I are the oscillating solutions of an electron in a constant potential with energy greater than the potential energy. In region III it will be assumed that the electron is free to move in some constant effective potential V_e after it has tunneled through the barrier so that the solution in that region will be an oscillating solution for a particle travelling to the right. Its propagation constant is then equal to

$$k_3^2 = \frac{2m}{\hbar^2}(E - V_e)$$

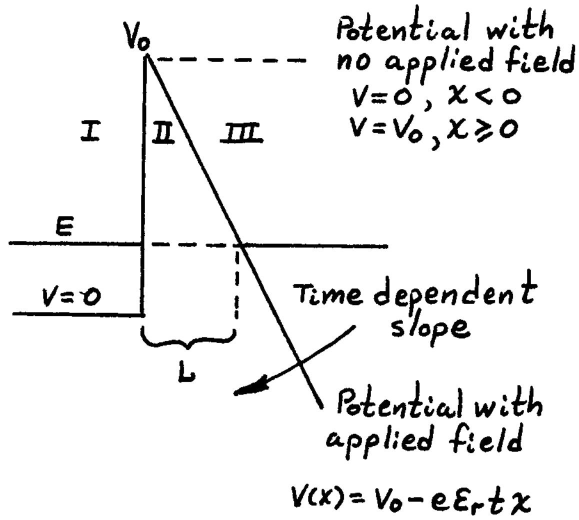

Fig 1. Potential barrier outside an electrode due to an applied electric field

In order to obtain the solution in region II, the time dependent Schroedinger equation must be solved. This is done according to the references (2, 3). The Hamiltonian is written as

$$H = H_o - F(t)X$$

$$F(t) = e\,\epsilon_r t, \qquad H_o = \frac{1}{2m}p^2 + V_o$$

If the state function at any time is given in the Schroedinger Picture as a state $\psi(t)$ developed from the initial state $\psi(o)$ by the time development operator U(t) according to the equation

$$\psi(t) = U(t)\,\psi(0)$$

it is found that the resulting operator equation cannot be simply solved for since the new Hamiltonian does not commute with itself at different times. However, by introducing a new time development operator $\underline{U}(t)$ according to the equation

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$$U(t) = e^{i\alpha(t)x}\ \underline{U}(t)$$

where $\alpha(t)$ is a c—number, it is found that the new Hamiltonian commutes with itself at different times, and then the complete time development operator can be written as

$$U(x,\ p,\ t) = e^{i\alpha x} \exp - \frac{i}{\hbar}\quad H_o t + \frac{\hbar}{m}\beta p + \frac{\hbar^2}{2m}\gamma$$

where

$$\alpha(t) = \frac{1}{\hbar}\int_o^t F(t')dt',\ \beta(t) = \int_o^t \alpha(t')dt',\ \gamma(t) = \int_o^t [\alpha(t')]^2 dt'$$

For the state functions at t = 0, the exponentially decaying and growing solutions, corresponding to an electron in a static potential V_o, are used so that the time dependent wave functions in region II take the form

$$\psi(t) = e^{(i\alpha \pm K)x\ +\ b_\pm}$$

where

$$b_\pm = \frac{i\hbar}{2m}(\kappa^2 t - \gamma) \mp \frac{\hbar}{m}\kappa\,\beta - \frac{i}{\hbar}V_o\, t$$

$$\kappa^2 = \frac{2m}{\hbar^2}(V_o - E)$$

The solutions for all three regions can now be written as

$$\psi_I = A\, e^{ik_1 x} + Be^{-ik_1 x}$$

$$\psi_{II} = Ce^{(i\alpha + K)x\ +\ b_+} + De^{(i\alpha - K)x\ +\ b_-}$$

$$\psi_{III} = Fe^{ik_3 x}$$

By matching the wave functions and their gradients at x = 0 and x = L, the ratio F/A can be solved for. The transmission coefficient can then be obtained as

$$T(E,\ t) = \frac{k_3}{k_1}\left|\frac{F}{A}\right|^2$$

$$= \frac{4k_1k_3\kappa^2}{[k_1k_3 - \kappa^2 + \alpha^2\ +\ \alpha(k_3 - k_1)]^2 \sinh^2\kappa L\ +\ \kappa^2(k_3+k_1)^2\cosh^2\kappa L}$$

Note that at t = 0, T(E) = 0. This is similar in form to the transmission coefficient for a rectangular barrier of width L in which $V_I = 0$, $V_{II} = V_o > E$, and $V_{II} = -V_3$ and given by

$$T(E)_r = \frac{4k_1k_3\kappa^2}{[k_1k_3 - \kappa^2]^2\sinh^2\kappa L\ +\ \kappa^2(k_1+k_3)^2\cosh^2\kappa L}$$

where

$$k_3^2 = \frac{2m}{\hbar^2}(V_o - V_3)$$

It is seen that a time dependence appears in a more explicit way than just through

$$L(t) = (V_o - E)/e\ \epsilon_r t$$

Time constants can be defined as

$$\tau_1^4 = 2\kappa\sqrt{k_1k_3}\ /\alpha_o^2$$

$$\tau_2^2 = 2\kappa\sqrt{k_1k_3}\ /\alpha_o\ (k_3 - k_1)$$

$$\tau_r = \kappa(V_o - E)/e\ \epsilon_r$$

$$\alpha_o = e\ \epsilon_r/2\hbar$$

so that the transmission coefficient can be written as

$$T(e,\ t) =$$

$$\frac{1}{\left[\dfrac{k_1k_3 - \kappa^2}{2\kappa\sqrt{k_1k_2}} + \left(\dfrac{t}{\tau_1}\right)^4 + \left(\dfrac{t}{\tau_2}\right)^2\right]^2 \sinh^2\left(\dfrac{\tau_r}{t}\right) + \dfrac{(k_3+k_1)^2}{4k_1k_3}\cosh^2\left(\dfrac{\tau_r}{t}\right)}$$

showing that there are three intrinsic time constants associated with the tunneling probability.

If it is assumed that the effective potential outside the metal is zero, then

$$k_3 = k_1 = k$$

Looking at T(E,t) at small times, it reduces to the expression for $T(E)_r$; namely

$$T(E) = \frac{16\ k^2\kappa^2 e^{-2\kappa L}}{(k^2 - \kappa^2)^2\ +\ 4k^2\kappa^2}$$

Comparing this to the Fowler-Nordheim expression (4)

$$T(E)_{FN} = \frac{4\sqrt{E}\sqrt{V_o - E}}{V_o}\ e^{-2\kappa L}$$

it is seen that

$$T(E, t \text{ small}) = (T(E)_{FN})^2$$

Since the probabilities are less than one, it is seen that

$$T(E, t) < T(E)_{FN}$$

This implies that the Fowler-Nordheim expression over estimates the probability that an electron will tunnel through the barrier.

It should be noted that as $t \to \infty$, the barrier width L goes to zero, and the electric field intensity increases without limit resulting in

$$T(E, t \to \infty) \to 0$$

This is a curious result since one would think that as $L \to 0$ the electron would see no barrier; the limit for the rectangular barrier ($k_3 = k_1$) as $L \to 0$ is one. Since in the limit

$$\lim_{t \to \infty} V(X) = V_o - e\,\epsilon_r tx) = -\infty$$

a better comparison would be with the rectangular barrier with $V_{III} = -V_3$ and letting $V_3 \to \infty$. From the above, it is seen that

$$\lim_{k_3 \to \infty} T(E)_r) = 0$$

With respect to the Fowler-Nordheim result, one cannot use the $T(E)_{FN}$ above and let the width $L \to 0$ because approximations were made (4) corresponding to low field intensities. If one uses the full expression derived by Fowler and Nordheim and takes the limit $\epsilon \to \infty$, one finds that

$$\lim_{\epsilon \to \infty} T(E)_{FN} = 0$$

This result points out a quantum mechanical aspect of the problem that must be accepted if a quantum mechanical interpretation is made of field emission; namely, for $t \to \infty$ ($\epsilon \to \infty$), not only is $L \to 0$, but an infinite discontinuity is developing from which total reflection occurs (the wave function at an infinite discontinuity vanishes (5)).

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